A Note on the Invariant Distribution of a Stochastic Dynamical System

Pengcheng Xie*

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Abstract

This paper demonstrates the invariant distribution of a stochastic dynamical system. We give the invariant distribution and numerical examples. We also present a further discussion on the computation details.

Keywords Invariant distribution · stochastic dynamical system

1 The stochastic dynamic system and invariant distribution

The stochastic dynamic system we focus on is

$$\begin{cases} dx = vdt \\ dv = -\nabla V dt - \gamma(v - Ax)dt + Avdt + \sigma d\omega, \end{cases} \tag{1.1}$$

where

$$V = \frac{1}{2}x^T Bx.$$

This equation can be written as

$$d\begin{pmatrix} x \\ v \end{pmatrix} = M\begin{pmatrix} x \\ v \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \sigma d\omega,$$

where

$$M = \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix}.$$

*Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China.
Email: xpc@lsec.cc.ac.cn; Corresponding author.

We can obtain the solution of the initial form of the problem is

$$x(t) = e^{Mt}x(0) + \int_0^t e^{M(t-\tau)} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} d\omega_\tau.$$

Remark. We assume that

$$\lim_{t\to+\infty} e^{Mt} = 0.$$

Then $x(t)$ will converge to a limit distribution when $t \to \infty$. It can be proved that it is a normal distribution $\mathcal{N}(0, C)$, where

$$C = \sigma^2 \int_0^{+\infty} \exp(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp\left(M^T t\right) dt.$$

Therefore, we obtain a linear system below

$$\begin{aligned} MC + CM^T &= \sigma^2 \int_0^{+\infty} \left(\text{Mexp}(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp\left(M^T t\right) + \exp(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp\left(M^T t\right) M^T \right) dt \\ &= \sigma^2 \int_0^{+\infty} \frac{d}{dt} \left(\exp(Mt) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \exp\left(M^T t\right) \right) dt \\ &= -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \end{aligned}.$$

Then we can obtain the theorem below

Theorem 1.1. *Suppose that we have an equation*

$$\begin{cases} dx = vdt \\ dv = -\nabla V dt - \gamma(v - Ax)dt + Avdt + \sigma d\omega. \end{cases} \tag{1.2}$$

Let

$$M = \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix}$$

Suppose that

$$\lim_{t\to+\infty} e^{Mt} = 0,$$

then the limit distribution is $\mathcal{N}(0, C)$ where

$$MC + CM^T = -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

From this equation we can not only solve the matrix C easily but also obtain some corollaries. Here we give the most obvious one.

Corollary 1.1. *Suppose that*

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}.$$

We have

(1) C_2 is a skew-symmetric matrix.

(2) the diagonal elements of C_2 are all 0.

Proof. From

$$MC + CM^T = -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

We can obtain

$$C_2 + C_3 = 0.$$

But C is the covariance matrix of a normal distribution, so C is symmetric. Therefore,

$$C_3 = C_2^T$$

We conclude

$$C_2^T = -C_2$$

So (1) is proved. (2) is a corollary of (1). □

2 Numerical examples

We give two examples.

Example 2.1.

$$n = 2, \sigma = 1, \gamma = 1, A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, B = I.$$

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & b\gamma & -\gamma & b \\ 0 & -1 & 0 & -\gamma \end{pmatrix},$$

$$C = \begin{pmatrix} c_{11} & c_{12} & 0 & c_{14} \\ c_{12} & c_{22} & -c_{14} & 0 \\ 0 & -c_{14} & c_{33} & c_{34} \\ c_{14} & 0 & c_{34} & c_{44} \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$MC + CM^T = D.$$

We obtain

$$c_{11}=\frac{1}{4}\left(3b^2+2\right),c_{12}=\frac{b}{2},c_{22}=\frac{1}{2},c_{33}=\frac{1}{2}\left(b^2+1\right),c_{34}=\frac{b}{4},c_{44}=\frac{1}{2},c_{14}=-\frac{b}{4}.$$

Therefore

$$C=\left(\begin{array}{cccc} \frac{1}{4}\left(3b^2+2\right) & \frac{b}{2} & 0 & -\frac{b}{4} \\ \frac{b}{2} & \frac{1}{2} & \frac{b}{4} & 0 \\ 0 & \frac{b}{4} & \frac{1}{2}\left(b^2+1\right) & \frac{b}{4} \\ -\frac{b}{4} & 0 & \frac{b}{4} & \frac{1}{2} \end{array}\right).$$

$$\rho=\frac{1}{\sqrt{|2\pi C|}}\exp\left(-\frac{1}{2}\left(q_1,q_2,p_1,p_2\right)C^{-1}\left(q_1,q_2,p_1,p_2\right)^T\right).$$

From the Fokker-Planck equation [1]

$$0=\frac{d\rho}{dt}=\sum_{i=1}^{n}-\frac{\partial}{\partial q_i}\left(p_i\rho\right)-\frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V}{\partial q_i}-\gamma\left(p_i-\sum_j A_{ij}q_j\right)+\sum_j A_{ij}p_j\right)\rho\right]+\frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho,$$

we can confirm this result.

Example 2.2.

$$n=1.A=a,B=b$$

$$M=\left(\begin{array}{cc} 0 & 1 \\ \gamma a-b & a-\gamma \end{array}\right)$$

From

$$MC+CM^T=\left(\begin{array}{cc} 0 & 0 \\ 0 & -\sigma^2 \end{array}\right),$$

we can obtain

$$C=\left(\begin{array}{cc} \frac{\sigma^2}{2(a-\gamma)(\gamma a-b)} & 0 \\ 0 & -\frac{\sigma^2}{2(a-\gamma)} \end{array}\right)$$

$$\rho=\frac{1}{\sqrt{|2\pi C|}}\exp\left(-\frac{1}{2}(q,p)C^{-1}(q,p)^T\right).$$

From the Fokker-Planck equation

$$0=\frac{d\rho}{dt}=\sum_{i=1}^{n}-\frac{\partial}{\partial q_i}\left(p_i\rho\right)-\frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V}{\partial q_i}-\gamma\left(p_i-\sum_j A_{ij}q_j\right)+\sum_j A_{ij}p_j\right)\rho\right]+\frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho,$$

3 The computation analysis

In this section, we take more discussion on

$$MC+CM^T=-\sigma^2\left(\begin{array}{cc} 0 & 0 \\ 0 & I \end{array}\right).$$

Assume that $C' = \frac{1}{\sigma^2}C$, and in this case, we obtain:

$$MC' + C'M^T = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

For conveniently writing, we denote C' as C in the following

$$MC + CM^T = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

where M is known, C is unknown based on simple calculation, we have the conclusion that the total account of the unknown variable is $2n \times 2n$. The system is equivalent to a linear equation set $DX = y$, where $d \in \Re^{4n^2 \times 4n^2}, x \in \Re^{4n^2}, y \in \Re^{4n^2}$. Since

$$C = \begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix},$$

where C_1 is a symmetric matrix, C_2 is a skew symmetric matrix, C_4 is a symmetric matrix, the amount of the unknown components, in fact, is only

$$\frac{1}{2}n(n+1) \times 2 + \frac{1}{2}n(n-1) = \frac{3}{2}n^2 + \frac{1}{2}n.$$

Then we obtain that

$$M = \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix},$$

$$\begin{aligned} MC + CM^T &= \begin{pmatrix} 0 & I \\ P & Q \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix} + \begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix}\begin{pmatrix} 0 & P^T \\ I & Q^T \end{pmatrix} \\ &= \begin{pmatrix} -C_2 & C_4 \\ PC_1 - QC_2 & PC_2 + QC_4 \end{pmatrix} + \begin{pmatrix} C_2 & C_2P^T + C_2Q^T \\ C_4 & -C_2P^T + C_4Q^T \end{pmatrix} \\ &= \begin{pmatrix} 0 & C_4 + C_1p^T + C_2Q^T \\ C_4 + PC_1 - QC_2 & PC_2 + QC_4 - C_2P^T + C_4Q^T \end{pmatrix} \\ &= -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \end{aligned}$$

where $P = \gamma A - B, Q = A - \gamma I$.

Thus we obtain that

$$\begin{cases} C_4 + C_1P^T + C_2Q^T = 0, \\ PC_2 + QC_4 - C_2P^T + C_4Q^T = I. \end{cases} \tag{3.1}$$

According to the first line of (3.1), it holds that

$$C_4 = -C_1P^T - C_2Q^T.$$

Besides, we know C_4 is symmetric matrix, which means $C_4^T = C_4$ Thus we obtain:

$$\begin{aligned} C_4 &= -C_1P^T - C_2Q^T \\ &= -\left(C_1P^T - C_2Q^T\right)^T \\ &= -PC_1 - QC_2^T \\ &= -PC_1 - QC_2. \end{aligned} \tag{3.2}$$

Substitute (3.2) into the left hand side of the second line of (3.1), and then we obtain that

$$PC_2 + Q\left(-C_1P^T\right) - C_2P^T + \left(PC_1 + QC_2\right)Q^T = -I,$$

and

$$PC_2 - PC_1Q^T - C_2P^T - QC_1P^T = -I.$$

Thus calculating the covariance matrix C which is equivalent to solving (3.1) is finally equivalent to solving the equations

$$\begin{cases} C_4 + C_1P^T + C_2Q^T = 0 \\ P\left(C_2 - C_1Q^T\right) + \left(C_2 - C_1Q^T\right)^T P^T = -I \end{cases} \tag{3.3}$$

without the integral.

Example 3.1.

$$n = 2, \sigma = 1, \gamma = 1, A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, B = I.$$

so

$$P = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}.$$

We denote C_1 by C_{ij}^1, C_2 by C_{ij}^2 and C_4 by C_{ij}^4. According to symmetry or skew-symmetry of C_1, C_2, C_4, we obtain the linear equations:

$$\begin{aligned} -2C_{11}^1 + 2bC_{12}^1 - 2C_{11}^2 + 2b\left(C_{21}^1 - bC_{22}^1 + C_{21}^2\right) &= -1, \\ -C_{12}^1 - C_{21}^1 + bC_{22}^1 - C_{12}^2 - C_{21}^2 + b\left(C_{22}^1 + C_{22}^2\right) &= 0, \\ -C_{11}^1 + bC_{12}^1 - C_{11}^2 + bC_{12}^2 + C_{11}^4 &= 0, \\ -C_{21}^1 + bC_{22}^1 - C_{21}^2 + bC_{22}^2 + C_{21}^4 &= 0, \\ -C_{12}^1 - C_{12}^2 + C_{12}^4 &= 0, \\ -C_{22}^1 - C_{22}^2 + C_{22}^4 &= 0, \\ -2C_{22}^1 - 2C_{22}^2 &= -1, \\ C_{12}^4 - C_{21}^4 &= 0, \\ C_{12}^1 - C_{21}^1 &= 0, \end{aligned}$$

$$
\begin{aligned}
C_{12}^2 + C_{21}^2 &= 0, \\
C_{11}^2 &= 0, \\
C_{22}^2 &= 0.
\end{aligned}
$$

Solving the equations, we obtain C as

$$
C = \begin{pmatrix}
\frac{1}{4}\left(2+3b^2\right) & \frac{b}{2} & 0 & -\frac{b}{4} \\
\frac{b}{2} & \frac{1}{2} & \frac{b}{4} & 0 \\
0 & \frac{b}{4} & \frac{1}{2}\left(1+b^2\right) & \frac{b}{4} \\
-\frac{b}{4} & 0 & \frac{b}{4} & \frac{1}{2}
\end{pmatrix}.
$$

Fokker-Planck Equation helps us to check the solution

$$
\frac{d\rho}{dt} = \sum_{i=1}^{n} -\frac{\partial}{\partial q_i}\left(p_i \rho\right) - \frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V}{\partial q_i} - \gamma\left(p_i - \sum_j A_{ij} q_j\right) + \sum_j A_{ij} p_j\right)\rho\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho = 0.
$$

4 Conclusions

This paper demonstrates the invariant distribution of a stochastic dynamical system. We give the invariant distribution and numerical examples. We also give the details of the computation analysis.

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References

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